SOME UPPER BOUNDS FOR THE DIAMETERS OF CONVEX POLYTOPES

BY

P. R. GOODEY

ABSTRACT

$\Delta(d, n)$ is defined to be the maximum diameter of a d-polytope with n-facets. The main results of the work are an evaluation of $\Delta(4, 10)$ and $\Delta(5, 11)$. Also, improved upper bounds are found for $\Delta(6, 13)$ and $\Delta(7,14)$.

A *d-polytope* is the convex hull of a finite set of points whose affine hull is E^d. The points form *vert P*, the set of *vertices* of the polytope P, and the faces of dimension $d-1$ are called the *faces*. A d-polytope is said to be *simple* if each vertex belongs to precisely d facets. If P is a simple d-polytope and $x \in \text{vert } P$ then the d facets through x are called the *x-facets*. If P is a d-polytope with n facets, we write $P \in (d,n)$. If x and y are two vertices of P then $\delta(x,y)$ denotes the combinatorial distance between x and y. The *diameter* $d(P)$ of P is defined by $d(P) = \max\{\delta(x,y) : x, y \in \text{vert } P\}$. Two vertices x, y of P are said to be *diametral vertices* if $\delta(x,y) = d(P)$. Finally, we write $\Delta(d,n) = \max\{d(P) : P \in (d,n)\}$; we note that this is the same as $\Delta_b(d,n)$ in [1].

The purpose of this work is to obtain improvements on the known estimates for $\Delta(d,n)$ for certain values of d and n. The main results are an evaluation of $\Delta(4,10)$ and $\Delta(5,11)$. We also improve the known upper bounds for $\Delta(6,13)$ and $\Delta(7,14)$. These results are primarily concerned with the d-step conjecture which can be stated as saying $\Delta(d,n) \leqq n-d$. Combining the work of Klee and Walkup [1] and Larman [2], the known results which we shall need are:

$$\Delta(3,n) = \left[\frac{2n}{3}\right] - 1,\ n \geqq 4:\ 5 \leqq \Delta(4,10) \leqq 6:$$

$$6 \leqq \Delta(5,11) \leqq 7:\ \Delta(4,11) \leqq 7:\ \Delta(5,12) \leqq 9:$$

$$\Delta(6,13) \leqq 10:\ \Delta(7,14) \leqq 11.$$

Received November 29, 1971

LEMMA. *If $P \in (3,9)$ and $d(P) = 5$ then,*

(i) *P is simple and has 14 vertices,*

(ii) *If a_0, $a_5 \in$ vert P and $\delta(a_0, a_5) = 5$ then $\delta(a_0, v) \leqq 4$ for all $v \in$ vert $P \backslash \{a_5\}$,*

(iii) *If $P_i = a_0, a_1^i, a_2^i, a_3^i, a_4^i, a_5$ $(i = 1,2,3)$ are 3 disjoint paths from a_0 to a_5 then the 12 vertices a_j^i $(i = 1,2,3,; j = 1,2,3,4)$ form 6 disjoint pairs $\{a_k^i, a_t^j\}, |k - t| \leqq 1$, $i \neq j$ such that (a_k^i, a_t^j) is an edge.*

(iv) *Each facet containing a_0 or a_5 has at most 6 vertices,*

(v) *If a facet containing a_0 (or a_5) is an n-gon with $n \geqq 5$ then the other two facets containing a_0 (or a_5) are a triangle and a quadrilateral,*

(vi) *There is at most one facet not containing a_0, a_1^1, a_1^2, a_1^3 and a_5. If such a facet F is present, then one of the facets that contain a_0 is a triangle. Further, if F has 6 or more vertices then it must contain two neighbours of a_5.*

vii) *P cannot have a triangular facet not containing a_0 or a_5.*

PROOF. The results of this lemma, apart from (vi) are due to Larman [2] or are immediate corollaries of his results, and so it will suffice to prove (vi).

Using (iv) and the fact that $\delta(a_1^i, a_5) = 4$ $(i = 1,2,3)$ one can see that the neighbours of a_0 lie on at least 2 distinct facets not containing a_0 or a_5. This means there is at most one remaining facet not containing a_0, a_5 nor any neighbour of a_0 since $P \in (3,9)$. If no facet containing a_0 is a triangle then the neighbours lie on 3 distinct facets not containing a_0 and a_5. Thus each facet of P contains either a_0, a_5 or a neighbour of a_0. Since the graph of P is planar and the P_i are distinct, exactly two of the P_i may meet the facet; also no P_i can use more than 3 vertices of the facet. This shows that the facet can be at most hexagonal and in that case it must contain two of the vertices a_4^1, a_4^2 and a_4^3.

We now proceed to the evaluation of $\Delta(4,10)$ and $\Delta(5,11)$.

THEOREM 1. $\Delta(4,10) = 5$.

PROOF. Following the results of Klee and Walkup [1], it suffices to show that if $P \in (4,10)$ is simple and x and y are vertices of P not having a facet in common then $\delta(x, y) \leqq 5$. To this end we suppose $\delta(x, y) = 6$ and seek a contradiction. This is sufficient since Larman [2] has shown that we must have $\delta(x, y) \leqq 6$.

Let the x-facets be $F_1, \cdots, F_4$; the y-facets $G_1, \cdots, G_4$ and the remaining facets X and Y. Let the neighbours of x be $x_1, \cdots, x_4$ and those of y be $y_1, \cdots, y_4$ such that $x_i \notin F_i$ and $y_i \notin G_i$, $i = 1, \cdots, 4$. We split the proof into two possibilities.

CASE 1. *A neighbour of x (or y) belongs to a y- (or x-) facet.*

We may suppose without loss of generality that $x_1 \in G_1$. Then since $\delta(x, y) = 6$ we must have $d(G_1) = 5$ and so $G_1 \in (3,9)$ with x_1 and y as diametral vertices. If $x_i \in G_1$ with $i \neq 1$, then by (ii), $\delta(y, x_i) \leqq 4$ and so $\delta(x, y) \leqq 5$. Thus we may assume that $x_i \notin G_1$ for $i = 2,3,4$. So no neighbour of x_1 in G_1 lies in F_1, since any such neighbour would be an x_i with $i \neq 1$. Since $G_1 \in (3,9)$, $G_1 \cap F_1 \neq \emptyset$. Therefore, by (vi), $G_1 \cap F_1$ is the facet of G_1 not containing y nor any neighbour of x_1, and we may assume without loss of generality that $F_3 \cap G_1$ is a triangle.

Let G_1' be the facet of P containing y_1 but not y. Then, since $G_1 \in (3,9)$, $G_1 \cap G_1' \neq \emptyset$. Assume $x_1 \in G_1' \cap G_1$, then $x \in G_1'$ and so $G_1' \in (3,9)$ since otherwise $\delta(x, y_1) < 5$. Hence applying (vii) to G_1'we see that $G_1 \cap G_1'$ is not a triangle. So we may put $G_1' = F_2$, and thus $d(F_2) = 5$ and $F_2 \in (3,9)$. Now $F_2 \cap G_1$ is a facet of F_2 with at least 4 vertices. Since $x \notin G_1$ and $x_i \notin G_1$ for $i \neq 1$, $F_2 \cap G_1$ must contain a vertex, α say, which is distance 3 from x and 2 from y_1. So α and y_1 have a common neighbour, α_1 say. Now $\alpha_1 \notin G_1$, since otherwise α_1 would be y or a neighbour of y both of which are impossible. If $\alpha_1 \in F_i$ with $i \neq 2$ then $\alpha_1 \in F_2 \cap F_i$ and so $F_2 \cap F_i$ is a facet of F_2 containing α_1 and x. But this is impossible by (iv) since $\delta(x, \alpha_1) = 4$. Now $y_1 = F_2 \cap G_2 \cap G_3 \cap G_4$ and $\alpha_1 \in F_2$, so α_1 belongs to exactly two of G_2, G_3, G_4. Thus α must belong to exactly one of G_2, G_3, G_4 since $\delta(\alpha, y) = 3$. So we may assume that $\alpha_1 \in F_2 \cap G_2 \cap G_3$ and $\alpha \in F_2 \cap G_1 \cap G_2$. Thus since $\delta(\alpha, y) = 3$, $G_1 \cap G_2$ must be a hexagonal facet of G_1 by (iv). Let the vertices of this facet be α, β_1, β_2, y, β_3, β_4 where $\beta_1 \in F_2 \cap G_1 \cap G_2$. Then since $\delta(y, x_1) = 5$ we must have $\delta(\beta_1, x_1) = 3$ and so, by (iv). $G_1 \cap F_2$ is also a hexagon. Now by (ii), no neighbour of y_1 in F_2 can lie in G_1. Thus, since $x \notin G_1$, we have the desired contradiction using (vi) and the fact that $x_i \notin G_1$ if $i \neq 1$.

So now we must assume that $x_1 \notin G_1'$. We note that we may assume $y_i \notin F_1$ for $i = 2,3,4$ since otherwise we would have a similar situation to that above. Also, because of (ii), no neighbour of x_1 in G_1 can lie in F_1. Thus, using (vi), (v) and the fact that G_1 has 14 vertices, we deduce that the facets of G_1 containing x_1 are a triangle, a quadrilateral and a pentagon and the same is true of the facets containing y. So the graph of G_1 must be one the two possibilities shown in Fig. 1 with $k = 1$ and $j = 2,3$ or 4.

If $y_1 \in F_1$, the two vertices of $F_1 \cap G_1 \cap G_j$ must be distance 4 from x and therefore must both be neighbours of y_1. But this is impossible since $y_1 =$

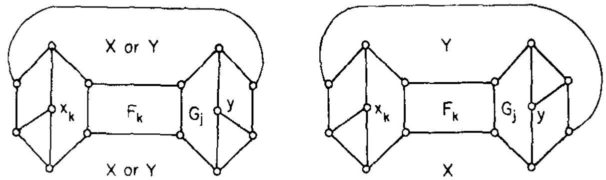

Fig. 1

$G_2 \cap G_3 \cap G_4 \cap F_1$. So now we know that $y_i \notin F_1$ for $i=1,\cdots,4$. We assume without loss of generality that $y_1 = G_2 \cap G_3 \cap G_4 \cap X$, $j=2$, $F_1 \cap G_1 \cap F_2 \neq \varnothing$ and $G_3 \cap G_1$ is a triangle. Then, if $x_4 \in X \cup Y$ then x_4 has a neighbour, v say, on G_1 such that $\delta(v, y) = 3$. But then we would have $\delta(x, y) = 5$, so this is impossible.

Also we know that $x_4 \notin G_1$. If $x_4 \in G_2$ then, by the arguments above, the graph of G_2 must be one of the possibilities shown in Fig. 1 with $k=4$ and $j = 1$, 3 or 4. Clearly we must have $j = 1$ by (v) since $G_2 \cap G_1$ is a pentagon. But from the graph of G_1, we see that $F_4 \cap G_1 \cap G_2 = \varnothing$, and hence $x_4 \notin G_2$. Similarly, we may deduce that $x_3 \notin X \cup Y \cup G_1 \cup G_2$. If $x_4 \in G_3$, then by (ii) we must have $x_3 \in G_4$ and the graph of G_3 must be one of the two possibilities shown in Fig. 1 with $k=4$ and $j = 1, 2$ or 4. Now $j \neq 1$ since from the graph of G_1, $G_1 \cap F_4 \cap G_3 = \varnothing$. If $j = 4$, then $F_4 \cap G_3 \cap G_4 \neq \varnothing$, but this is inconsistent with the graph of G_4. So we must have $j = 2$ and since $G_3 \cap G_1$ is a triangle, we have $G_3 \cap G_4$ is a quadrilateral. Thus $G_1 \cap G_4$ and $G_3 \cap G_4$ are both quadrilaterals containing y, but since $x_3 \in G_4$, there can only be one quadrilateral facet of G_4 containing y. Hence $x_4 \notin G_3$. Since the above argument is symmetric in x_3 and x_4, we deduce that $x_3 \notin G_3$ and so we have $x_3 \in G_4$ and $x_4 \in G_4$. But then $\delta(x, y) \leqq 5$, by (ii). This completes the proof of Case I.

Case II. No neighbour of x *(or* y*) belongs to a* y*-(or* x*-) facet.*

All the neighbours of x and y belong to either X or Y. Clearly we cannot have all the neighbours of x belonging to X.

Assume $x_i \in X$ for $i = 1,2,3$ and $x_4 \in Y$. Then if $y_j \in X$ for some j, we must have $\delta(x_i, y_j) \geqq 4$ for $i = 1,2,3$. We deduce that there can be only one such value of j. If there were three such values of j, say $j = 1,2,3$, then X would have the two triangular facets $F_4 \cap X$ and $G_4 \cap X$. Replacing these facets by single vertices results in a 3-polytope with at most 7 facets and which must have diameter 4. This is impossible and so we assume j takes the two values 1 and 2. Then $F_4 \cap X$ is a triangular facet of X, and y_1 and y_2 are neighbours in X. We remove the triangular facet replacing it by a single vertex, v say. This gives us a 3-polytope X' with at most 8 facets and vertices v, y_1 and y_2 such that $\delta(v, y_i) = 4$ for $i = 1,2$. Hence X' must have 8 facets and 12 vertices. The vertices must lie on 3 independent paths from v to y, two of which are of length 4 and the other of length 5. Then we note that X' must have a triangular facet containing y_1 and y_2. But the third vertex of this triangle must be a neighbour of y, contradicting the assumption that there are only two possible values of j. This shows that there is only one possible value of j. If there is such a value we note that X must have 9 facets since we can again reduce it to a 3-polytope X' of diameter 4. So now we may assume further that $y_i \in Y$, $i = 2,3,4$ and $y_1 \in X$. Thus since $\delta(x, y) = 6$, Y must also have 9 facets. Now F_4 is a tetrahedron with facets $F_4 \cap F_i$, $i = 1,2,3$, and $F_4 \cap X$. Thus $F_4 \cap Y = \varnothing$ and Y cannot have 9 facets. Therefore, we have shown that X cannot contain 3 neighbours of x.

Because of the symmetry of the problem, we may finally assume that $x_i \in X$, $y_i \in X$ for $i = 1,2$ and $x_i \in Y$, $y_i \in Y$ for $i = 3,4$. As before, x_1 and x_2 cannot have a neighbour in common in X and the same is true of y_1 and y_2. Now if $\delta(x_i, y_i) = 4$ for $i = 1,2$, then let $x_i \gamma_2^i \gamma_3^i \gamma_4^i \, y_i$ be a path of length 4 from x_i to y_i for $i = 1,2$. Then $\gamma_2^1 \neq \gamma_2^2$, $\gamma_4^1 \neq \gamma_4^2$, $\gamma_2^1 \neq x_2$, $\gamma_1^2 \neq x_1$, $\gamma_4^1 \neq y_2$ and $\gamma_4^2 \neq y_1$. Also we note that $\gamma_3^1 \neq \gamma_3^2$. Thus these vertices together with the four remaining neighbours x_1, x_2 and y_1, y_2 are the 14 vertices of X. But it is clear that the graph of X cannot be 3-connected. So we may assume that $\delta(x_1, y_1) = 5$. In this case we must have $\delta(x_2, y_2) = 3$ as X does not have a triangular facet containing the pair x_1, x_2 or the pair y_1, y_2. This completes the proof of the theorem.

COROLLARY. $\Delta(5,11) = 6$

PROOF. Following the results of Klee and Walkup [1], it suffices to show that if $P \in (5,11)$ is simple and x and y are vertices of P not sharing a facet then $\delta(x, y) \leqq 6$.

Let the x-facets be $F_1, \cdots, F_5$; the y-facets be $G_1, \cdots, G_5$ and the remaining facet X. Then there is at least one neighbour of x, x_1 say, which does not lie in X. So $x_1 \in G_j$ for some j. Now $G_j \in (4, n)$ with $n \leqq 10$ and so, from Theorem 1, $d(G_j) \leqq 5$. Hence $\delta(x, y) \leqq 6$.

We note that this corollary constitutes a proof of the 6-step conjecture for 5-polytopes. Also it is now trivial to deduce the result of Larman [2] that $\Delta(6,12) \leqq 7$. Next we obtain some improvements of results concerned with the 7-step conjecture.

THEOREM 2. *If* $P \in (5,12)$ *and* x *and* y *are vertices of* P *such that* $\delta(x, y) = 9$, *then* $\delta(y, v) = 8$ *for any neighbour* v *of* x.

PROOF. Again we may assume that the x-facets are $F_1, \cdots, F_5$, the y-facets are $G_1, \cdots, G_5$ and the remaining facets are X and Y. Since $\delta(x, y) = 9$ and $\Delta(4,11) \leqq 7$, no neighbour of x (or y) can lie in a y-(or x-) facet. Assume v is a neighbour of x with $\delta(y, v) = 9$ and assume $v \in X$. Then no neighbour of y can lie in X. Hence all neighbours of y must lie in Y which is clearly impossible. This completes the proof of the theorem.

THEOREM 3. $\Delta(6,13) \leqq 9$.

PROOF. We let $P \in (6,13)$, assume x and y are vertices of P such that $\delta(x, y) = 10$ and seek a contradiction.

Let the x-facets be $F_1, \cdots, F_6$; the y-facets $G_1, \cdots, G_6$ and the remaining facet X. Let the neighbours of x be x_i with $x_i \notin F_i$, $i = 1, \cdots, 6$ and those of y be y_i with $y_i \notin G_i$, $i = 1, \cdots, 6$. There must be a neighbour of x which lies in a y-facet so we may assume that $x_1 \in G_1$ and $\delta(x_1, y) = 9$. If $x_i \in G_1$ for some $i \neq 1$ then x_1 and x_i are neighbours in G_1. So by Theorem 2, $\delta(y, x_i) = 8$ and thus $\delta(x, y) = 9$. Hence we may assume that $x_i \notin G_1$, $i = 2, \cdots, 6$ and thus no neighbour of x_1 in G_1 can lie in F_1. But not all neighbours of x_1 in G_1 lie in $X \cap G_1$ and so x_1 has a neighbour, v say, lying in G_j for some $j \neq 1$. Thus $v \in G_j \cap G_1$ and $y \in G_j \cap G_1$. Also $G_j \cap G_1 \in (4, n)$ where $n \leqq 11$ and so $\delta(v, y) \leqq 7$. Thus $\delta(x, y) \leqq 9$ and the proof is completed.

An immediate corollary of this result is $\Delta(7,14) \leqq 10$.

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DEPARTMENT OF MATHEMATICS
ROYAL HOLLOWAY COLLEGE
ENGLEFIELD GREEN, SURREY